IV. *A second Letter from Mr.* Colin Mc Laurin, *Professor of Mathematicks in the University of* Edinburgh *and F.* R. S. *to* Martin Folkes, *Esq; concerning the Roots of Equations, with the Demonstration of other Rules in Algebra; being the Continuation of the Letter published in the* Philosophical Transactions, Nº 394.

Edinburgh, April 19th, 1729.

S I R,

IN the Year 1725, I wrote to you that I had a Method of demonstrating Sir *Isaac Newton*'s Rule concerning the impossible Roots of Equations, deduced from this obvious Principle, that the Squares of the Differences of real Quantities must always be positive; and some time after, I sent you the first Principles of that Method, which were published in the *Philosophical Transactions* for the Month of *May*, 1726. The Design I have for some Time had of publishing a Treatise of Algebra, where I proposed to treat this and several other Subjects in a new Manner, made me think it unnecessary to send you the remaining Part of that Paper. But some Reasons have now determined me to send you with the Continuation of my former Method, a short Account of two other Methods in which I have treated the same Subject, and some Observations on Equations that I take to be new, and which will, perhaps, be more acceptable to you than what relates to the imaginary Roots themselves. Besides Sir *Isaac Newton*'s Rule, there arises from the following general

ral

ral Propofitions, a great Variety of new Rules, different from his, and from any other hitherto publifhed, for difcovering when an Equation has imaginary Roots. I fhall particularly explain one that is more ufeful for that Purpofe, than any that have been hitherto publifhed.

Suppofe there is an Equation of (n) Dimenfions of this Form,

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4}$$
$$- Ex^{n-5} + Fx^{n-6} - Gx^{n-7} + Hx^{n-8} -$$
$$Ix^{n-9} + Kx^{n-10} \&c. = 0.$$

And that the Roots of this Equation are, $a, b, c, d, e, f, g, h, i, k, l$, &c. then fhall $A = a + b + c + d + e + f$ &c. and therefore I call a, b, c, d, e, f, &c. *Parts* or *Terms* of the Coefficient A. For the fame Reafon I call $ab, ac, ad, ae, bc, bd, cd$, &c. Parts or Terms of the Coefficient B; abc, abd, abe, acd, bcd, &c. Parts or Terms of C; $abcd, abce, abcf$, Parts or Terms of the Coefficient D, and fo on. By the *Dimenfions* of any Coefficient; I mean the Number of Roots or Factors that are multiplied into each other in its Parts, which is always equal to the Number of Terms in the Equation that preceed that Coefficient. Thus A is a Coefficient of one Dimenfion, B of two, C of three, and fo of the reft. I call a Part or Term of a Coefficient C *fimilar* to a Part or Term of any Coefficient G, when the Part of G involves all the Factors of the Part of C : Thus $abc, abcdefg$ are fimilar Parts of C and G; after the fame manner $abcd, abcdef$ are fimilar Parts of D and F, the Part of F involving all the Factors of the Part of D. Thofe I call *diffimilar* Parts that involve no common Root or Factor: Thus abc, and $defgh$ are diffimilar Parts of the Coefficients C and F. The Sum of all
the

the Products that can be made by multiplying the Parts of any Coefficient C by all the fimilar Parts of G, I exprefs by $C'G'$ placing a fmall Line over each Coefficient : After the fame manner $D'F'$ expreffes the Sum of all the Products that can be made by multiplying the fimilar Parts of D and F by each other; and $C'\times C'$ expreffes the Sum of the Squares of the Parts of the Coefficient C, but $C'\times C$; expreffes the Sum of the Products that can be made by multiplying any two Parts of C by one another. Thefe Expreffions being underftood, and the five Propofitions in *Phil. Tranf.* N° 394, being premifed, next follows

PROP. VI.

If the Difference of the Dimenfions of any two Coefficients C and G be called (m) *then fhall the Product of thefe Coefficients multiplied by one another be equal to* $C'G' + \overline{m+2} \times B'H' + \dfrac{\overline{m+3}}{1} \times$

$\dfrac{m+4}{2} A'I' + \dfrac{m+4}{1} \times \dfrac{m+5}{2} \times \dfrac{m+6}{3} \times I \times K.$

Where B and H are the Coefficients adjacent to the Coefficients C and G, A and I the Coefficients adjacent to B and H, I and K the Coefficients adjacent to B and H.

It is known that $C = abc + abd + abe + abf + abg$, &c. and $G = abcdefg + abcdefh + abcdefi + bcdefgh$, &c. and it is manifeft,

1. That in the Product CG each Term of $C'G'$ will arife once as $a^2 b^2 c^2 defg$. But

2. Any Term of $B'H'$ as $a^2 b^2 cdefgh$ may be the Product of abc, and $abdefgh$, or of abd and $abcefgh$, or of abe and $abcdfgh$, or of abf and $abcdegh$,

$a\,b\,c\,d\,e\,g\,h$, or of $a\,b\,g$ and $a\,b\,c\,d\,e\,f\,h$, or laftly of $a\,b\,h$ and $a\,b\,c\,d\,e\,f\,g$; fo that it may be the Product of any Term of C that involves with $a\,b$ one of the Roots, c,d,e,f,g,h, multiplied by that Term of G, which involves ab and the other five; that is, it may arife in the Product C G as often as there are Roots in $a^2b^2c\,d\,e\,f\,g\,h$ befides a and b, or in general, as often as there are Units in the Difference of the Dimenfions of B and H, that is, $m + 2$ times ; becaufe m expreffes the Difference of the Dimenfions of C and G, and confequently in expreffing the Value of C G the Coefficient of the fecond Term B′ H′ muft be $m + 2$.

3. Any Term of A I, as $a^2b\,c\,d\,e\,f\,g\,h\,i$, may be the Product of any Part of C that involves the Root a with any two of the reft b,c,d,e,f,g,h,i (the Number of which is the Difference of the Dimenfions of A and I, which is in general equal to $m + 4$) multiplied by the Part of G that involves a and the other fix ; and therefore $a^2b\,c\,d\,e\,f\,g\,h\,i$ or any other Term of A′ I′ muft arife as often as different Products of two Quantities can be taken from Quantities whofe Number is $m + 4$, that

is $\overline{m + 4} \times \dfrac{\overline{m + 4} - 1}{2}$ times or $\dfrac{\overline{m + 3}}{1} \times \dfrac{\overline{m + 4}}{2}$

times ; and confequently in expreffing the Value of C G the Coefficient of the third Term A′ I′ muft be $\dfrac{\overline{m + 3}}{1}$ $\times \dfrac{\overline{m + 4}}{3}$.

4. Any Term of I × K as $a\,b\,c\,d\,e\,f\,g\,h\,i\,k$, may be the Product of any Part of C that involves three of its Factors, and of the Part of G that involves the reft, and therefore may arife in the Product C G as often as different

Pro-

Products of three Quantities can be taken out of Quantities whose Number is $m + 6$ that is, $\overline{m+6} \times \dfrac{m+5}{2}$

$\times \dfrac{m+4}{3}$ times, and therefore the Coefficient of the

fourth Term in the Value of C G muſt be $\dfrac{m+4}{1} \times$

$\dfrac{m+5}{2} \times \dfrac{m+6}{3}$.

In general, in expreſſing the Value of the Product of any two Coefficients C and G, if x expreſs the Order of any Term of this Value as $A' l'$, that is, the Number of Terms that precede it, the Coefficient of

that Term muſt be $\dfrac{2x+m}{1} \times \dfrac{2x+m-1}{2} \times$

$\dfrac{2x+m-2}{3}$ &c. taking as many Factors as there are

Units in x.

Cor. I. If it is required to find by this Propoſition the Square of any Coefficient E, then ſuppoſe $m = 0$, the Difference of the Dimenſions of the Coefficients in this Caſe vaniſhing, and we ſhall have $E^2 = E' \times E' +$

$2 \, D' \, F' + 3 \times \dfrac{4}{2} \times C' \, G' + 4 \times \dfrac{5}{2} \times \dfrac{6}{3} \times \, B' \, H'$

&c. $= E' \times E' + 2 \, D' \, F' + 6 \, C' \, G' + 20 \, B' \, H'$ $+ 70 \, A' \, l' + 252 \, K.$ Therefore if $E' \times E_{\prime}$ expreſs the Sum of the Products of any two parts of E multiplied by each other, we ſhall have $E^2 = E' \times E'$ $+ 2 \, E' \times E_{\prime}$, and therefore $E' \times E_{\prime} = D' \, F' +$ $3 \, C' \, G' + 10 \, B' \, H' + 35 \, A' \, l' + 126 \, K.$

Cor.

CoR. II. It follows from this Propofition that

$E^2 = E'\times E' + 2 D'F' + 6 C'G' + 20 B'H' + 70 A'I' + 252 K.$

$DF = \quad - \quad - \quad D'F' + 4 C'G' + 15 B'H' + 56 A'I' + 210 K.$

$CG = \quad - \quad - \quad - \quad - \quad C'G' + 6 B'H' + 28 A'I' + 120 K$

$BH = \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad B'H' + 8 A'I' + 45 K$

$AI = \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad A'I' + 10 K$

$K = \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad K.$

CoR. III. It eafily appears by comparing the Theorems given in the laft Corollary, that

$E'E' = \quad - \quad E^2 - 2 DF + 2 CG - 2 BH + 2 AI - 2 K.$

$D'F' = \quad - \quad - \quad - \quad DF - 4 CG + 9 BH - 16 AI + 25 K$

$C'G' = \quad - \quad - \quad - \quad - \quad - \quad CG - 6 BH + 20 AI - 50 K$

$B'H' = \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad BH - 8 AI + 35 K$

$A'I' = \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad AI - 10 K.$

P R O P. VII.

Let $l = n \times \dfrac{n-1}{2} \times \dfrac{n-2}{3}$ &c. *taking as many Factors as the Coefficient* E *has Dimenfions and* $\dfrac{l-1}{2l} \times$ E^2 *fhall always exceed* $D F - C G + B H - A I + K$ *when the Roots of the Equation are all real Quantities.*

For it is manifeft that l expreffes the Number of Parts or Terms in the Coefficient E, and it is plain from Propofition V (*See Phil. Tranf.* N° 394) that $\dfrac{l-1}{2l} \times E^2$ muft always be greater than the Sum of the Products that can be made by multiplying any two

of

of the Parts of E by each other, that is, than $E' \times E_{\prime}$; but $2 E' \times E_{\prime} = E^2 - E' E' = $ (by the firft Theo- rem in the laft Corollary) $2 DF - 2 CG + 2 BH$

$- 2 A' I + 2 K$, and therefore fince $\dfrac{l-1}{2l} \times E^2$

muft always exceed $E' E_{\prime}$, it follows that $\dfrac{l-1}{2l} E^2$

muft always be greater than $DF - CG + BH - AI + K$ when the Roots of the Equation are real Quantities.

SCHOL. In following my Method this was the firft general Propofition prefented it felf. For having firft ob- ferved that if l exprefes the Number of any Quantities,

the Square of their Sum multiplied by $\dfrac{l-1}{2l}$ muft

always exceed the Sum of the Products made by multi- plying any two of them by each other; and that the Ex- cefs was the Sum of the Squares of the Differences of the Quantities divided by $2l$, it was eafy to fee in the Equation $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4}$ &c. $= 0$. Since B is the Sum of the Pro- ducts of any two of the Parts of A, that if l exprefes

theNumber of the Roots of the Equation, $\dfrac{l-1}{2l} \times A^2$

muft always exceed B; and this is one Part of the 5th Propofition. In the next Place, I compared the Sum of the Products of any two Parts of B with AC, and found that it was not equal to AC but to AC — D from which I inferred, that if l exprefes

the

the Number of the Parts of B then $\dfrac{l-1}{2l} \times B^z$ muſt always exceed $AC - D$; and theſe eaſily ſuggeſted this general Propoſition.

P R O P. VIII.

Let r *expreſs the Dimenſions of the Coefficient* C_r, *and* s *the Difference of the Dimenſions of the Coefficients* C *and* G, *then* B *and* H *being Coefficients adjacent to* C *and* G, $\overline{n-r-s} \times r\,C'\,G'$ *ſhall always be greater than* $\overline{s+1} \times \overline{s+2} \times B'H'$ *when the Roots of the Equation are all real Quantities affected with the ſame Sign.*

For taking the Differences of all thoſe Parts of the Coefficient C that are ſimilar in all their Factors but one, as *a b c, a b h, a b i,* &c. and multiplying the Square of each Difference by ſuch Parts of the Coefficient D (which is of *s* Dimenſions) as are diſſimilar to both the Parts of C in that Difference, the Sum of all thoſe Squares thus multiplied, will conſiſt of Terms of C'G' taken poſitively, and of Terms of B'H' taken negatively. By multiplying in this manner $\overline{abc - abh}|^2 +$ $\overline{abc - abi}|^2 + \overline{abc - abk}|^2$ &c. $+ \overline{abc - ach}|^2 +$ $\overline{abc - aci}|^2 + \overline{abc - ack}|^2$ &c. $+ \overline{abc - bch}|^2 +$ $\overline{abc - bce}|^2 + \overline{abc - bck}|^2$ &c. by *d e f g* the Term of D, that is diſſimilar to all thoſe Parts of C, you will find that $a^2b^2c^2defg$ will ariſe in the Sum of the Products $r \times \overline{n - r - s}$ times : For thoſe Products may be alſo expreſſed thus $defg\,a^2b^2 \times \overline{c - h}|^2 + \overline{c - i}|^2 + \overline{c - k}|^2$ &c. $+ defg\,a^2c^2 \times \overline{b - h}|^2 + \overline{b - i}|^2 + \overline{b - k}|^2$ &c. $+$

$$defg\,b^2c^2$$

$d^ce f g b^2c^2 \times \overline{a-h}|^2 + \overline{a-1}|^2 + \overline{a-k}|^2$ &c. where the Number of the Differences $c-h$, $c-i$, $c-k$, &c. whose Squares are multiplied by $def g . a^2b^2$ is manifeftly equal to the Number of the Roots of the Equation that do not enter $a^2b^2c^2 def g$ or $a b c d ef g$, that is, to the Excefs of the Number of the Roots of the Equation above the Dimenfions of $a b c d ef g$, a Term of G, that is to $n-r-s$. But in collecting all the faid Products, $n-r-s \times a^2b^2c^2 def g$ muft arife as often as there are Units in r: Becaufe the Terms which are fubtracted from $a b c$ may differ from it in the Root c, as $a b h$, $a b i$, $a b k$, &c. or in the Root b, as $a c h$, $a c i$, $a c k$, &c. or in the Root a as $b c h$, $b c i$, $b c k$; that is, $n-r-s \times a^2b^2c^2 def g$ muft arife as often as there are Dimenfions in $a b c$, a Term of C, or as often in general as there are units in r, which expreffes the Dimenfions of C : Therefore the Term $a^1b^2c^2 def g$ will arife in the Sum of the above-mentioned Products $r \times \overline{n-r-s}$ times.

The Negative Part muft confift of the Terms of $B'H'$ doubled; each of which, as $2 a^2b^2c def g$ may arife as often as there can be Differences $c-d$, $c-e$, $c-f$, $c-g$, $d-e$, &c. affumed amongft the Terms c, d, e, f, g whose Number is equal to $s+2$ that is, $s+2$ $\times \dfrac{s+1}{2}$ times; and therefore $a^2b^2c def g$ or any other Part of $B'H'$ muft arife in the negative Part $\overline{s+1}$ $\times \overline{s+2}$ times ; and fince the whole aggregate muft be pofitive it follows $\overline{n-r-s} \times r \, C'G'$ muft always exceed $\overline{s+1} \times \overline{s+2} \times B'H'$.

<div align="right">Cor.</div>

Co r. I. Suppose we are to compare E′ E′ the Sum of the Squares of the Parts of E with D′ F′ the Sum of the Products of the similar Parts of D and F ; in this Case s vanishes, and therefore $\overline{n-r} \times r\, E' E'$ must exceed $2\, D' F'$. Let $\overline{n-r} \times r = m$ and consequently

$$\overline{n-r-1} \times \overline{r-1} = m - n + 1;\ \overline{n-r-2} \times$$
$$\overline{r-2} = m - 2n + 4;\ \overline{n-r-3} \times \overline{r-3} =$$
$$m - 3n + 9;\ \overline{n-r-4} \times \overline{r-4} = m - 4n + 16.$$

Since it is plain that $\overline{n-r-q} \times \overline{r-q} = \overline{n-r} \times r$ $- qn + q^2.$ Then by this Proposition, supposing

$$\overline{m \times \mathrm{E'E'} - 2\, \mathrm{D'F'}} = a'$$
$$\overline{m - n + 1} \times \mathrm{D'F'} - 12\, \mathrm{C'G'} = b'$$
$$\overline{m - 2n + 4} \times \mathrm{C'G'} - 30\, \mathrm{B'H'} = c'$$
$$\overline{m - 3n + 9} \times \mathrm{B'\, H'} - 56\, \mathrm{A'I'} = d'$$
$$\overline{m - 4n + 16} \times \mathrm{A'I'} - 90\, \mathrm{K'} = e'$$

The Quantities a', b', c', d', e', must be always positive when the Roots of the Equation are real Quantities affected with the same Sign. The Coefficients prefixed to the negative Parts are the Numbers 2,12,30,56,90, whose Differences equally increase by the same Number 8.

Co r. II. Supposing as before, that $\overline{n-r} \times r = m$; and also that $m \times \overline{m-n+1} = m';\ m' \times \overline{m-2n+4}$ $= m'';\ m'' \times \overline{m-3n+9} = m'''$ &c. it may be demonstrated after the manner of this Proposition, that if

$$m\, \mathrm{E'E'} - 2\, \mathrm{D'F'} = a'$$
$$m'\mathrm{E'E'} - 2 \times 12\, \mathrm{C'G'} = a''$$

$$m''\, \mathrm{E'E'} - 2 \times 12 \times 30\, \mathrm{B'H'} = a'''$$
$$m'''\, \mathrm{E'E'} - 2 \times 12 \times 30 \times 56\, \mathrm{A'I'} = a'''' \ \&c.$$

Then

Then fhall a', a'', a''', &c. be always pofitive when the Roots are real Quantities, whether they be affected with the fame, or with different Signs. The Negative Coefficients arife by multiplying thofe in the preceeding Corollary, 2,12,30,56,90, by one another.

P R O P. IX.

Let a',b',c',d',e', *and* m *exprefs the fame Quantities as in the Corollaries of the laft Propofition, and* m E^2—

$$\overline{m + n + 1} \times D F = a' + b' + 2c' + 5d' + 14e'.$$

For by Cor. ii Prop. vi.

$E^2 = E'E' + 2 D'F' + 6 C'G' + 20 B'H' + 70 A' I' + 252 K,$ and by the fame

$DF = - - - D'F' + 4 C'G' + 15 B'H' + 56 A'I' + 210 K;$

therefore $m E^2 - \overline{m + n + 1} \times D F = m E'E' +$

$\overline{m - n - 1} \times D'F' + \overline{m - 2n - 2} \times 2 C'G'$

$+ \overline{m - 3n - 3} \times 5 B'H' + \overline{m - 4n - 4} \times$

$14 A'I' + \overline{m - 5n - 5} \times 42 K =$ (by fubftituting fucceffively for $m E'E'$, $\overline{m - n + 1} \times D'F'$,

$\overline{m - 2n + 4} \times C'G'$, $\overline{m - 3n + 9} \times B'H'$,

$\overline{m - 4n + 16} \times A'I'$ their Values deduced from the firft Corollary of the laft Propofition) $= a' + b' +$ $2c' + 5d' + 14e'$, where the Coefficients prefixed to a',b',c',d',e', are the Differences of the Coefficients of E' E', D' F', C' G', B' H', A' I' and K in the Values of E^2 and D F taken from Cor. ii. Prop. vi. being 1 — 0, 2 — 1, 6 — 4, 20 — 15, 70 — 56 and 252 — 210.

Cor. Since $m = \overline{n - r} \times r$ therefore $m + n + 1$

$= \overline{n - r + 1} \times \overline{r + 1}$; and consequently $\dfrac{n - r}{n - r + 1} \times \dfrac{r}{r + 1} \times$

$\times E^r$ muſt always be greater than D F the Product of the Coefficients adjacent to E ; and hence the Fractions are deduced, that in Sir *Iſaac Newton*'s Rule are placed over the Terms of the Equation, which multiplied by the Square of the Terms under them, muſt always exceed the Products of the adjacent Terms of the Equation, when the Roots are real Quantities : For it is manifeſt that the Fraction to be placed over the Term $E x^{n-r}$ according to that Rule is the Quotient of $\dfrac{n - r}{r + 1}$ divided by $\dfrac{n - r + 1}{r}$.

P R O P. X.

The ſame Expreſſions being allowed as in the preceeding Propoſitions, it will be found in the ſame manner that as $m E^z - \overline{m + n + 1} \times D F = a' + b' + 2c' + 5d' + 14e'$ ſo

$m - \overline{n + 1} \times D F - \overline{m + 2n + 4} \times C G = \ - \ b' + 3c' + 9d' + 28e'$

$m - 2n + 4 \times C G - \overline{m + 3n + 9} \times B H = \ - \ - \ c' + 5d' + 20e'$

$m - 3n + 9 \times B H - \overline{m + 4n + 16} \times A I = \ - \ - \ - \ - \ d' + 7e'$

$m - 4n + 16 \times A I - \overline{m + 5n + 25} \times K = \ - \ - \ - \ - \ - \ - e.$

Theſe Theorems are eaſily deduced from the Theorems given in the ſecond Corollary of Prop. vi. and the firſt Corollary of the viii[th] Propoſition; and the Coefficients prefixed to $a', b', c', d', e',$ are the Differences of the Coefficients of the correſponding Terms in the Values of E^z, D F, C G, B H, A I and K in Cor. ii. Prop. vi.

Cor.

CoR. Hence the Products of any two Coefficients, as D F and A I may be compared together when the Sum of the Dimenſions of D and F is equal to the Sum of the Dimenſions of A and I. Let the Dimenſions of A and F be equal to s and m reſpectively, and

let $p = \dfrac{n-s}{s+1} \times \dfrac{n-s-1}{s+2} \times \dfrac{n-s-2}{s+3}$ &c, tak-

ing as many Factors as there are Units in the Difference

of the Dimenſions of D and A. Let $q = \dfrac{n-m}{m+1} \times$

$\dfrac{n-m-1}{m+2} \times \dfrac{n-m-2}{m+3}$ &c. taking as many Fact-

ors as you took in the Value of p. Then ſhall $\dfrac{q}{p} \times$

D F always exceed A I when the Roots of the Equation are real Quantities affected with the ſame Sign; and this Rule obtains, though the Roots are affected with different Signs when the Coefficients D and F are equal.

P R O P. XI.

The ſame Things being ſuppoſed as in the preceed-ing Propoſitions.

1. $m E^2 - \overline{m+1} \times 2\,DF + \overline{m+4} \times 2\,CG - \overline{m+9} \times$
   $2\,BH + \overline{m+16} \times 2\,AI - \overline{m+25} \times 2\,K \quad - - \Big\} = a'$.

2. $\overline{m-n+1} \times DF - \overline{m-n+4} \times 4\,CG + \overline{m-n+9} \times$
   $9\,BH - \overline{m-n+16} \times 16\,AI + \overline{m-n+25} \times 25\,K \Big\} = b'$.

3. $\overline{m-2n+4} \times CG - \overline{m-2n+9} \times 6\,BH + \overline{m-2n+16} \times$
   $20\,AI + \overline{m-2n+25} \times 50\,K \quad - - - - - - \Big\} = c'$.

4. $\overline{m-3n+9} \times BH - \overline{m-3n+16} \times 8\,AI + \overline{m-3n+25} \times 35\,K = d'$.

5. $\quad - - \quad \overline{m-4n+16} \times AI - \overline{m-4n+25} \times 10\,K = e'$.

L

Theſe

Thefe Theorems follow eafily from the third Corollary of the vi[th] Propofition. The firft eafily appears thus, $a' = m E' E' - 2 D' F' =$ (by that Corollary)

$m E^2 - 2 m D F + 2 m C G - 2 m B H + 2 m A I - 2 m K.$

$- 2 D F + 8 C G - 18 B H + 32 A I - 50 K.$

$= m E^2 - \overline{m + 1} \times 2 D F + \overline{m + 4} \times 2 C G - \overline{m + 9} \times 2 B H + \overline{m + 16} \times 2 A I - \overline{m + 25} \times 2 K.$

The other Theorems are deduced from the fame Corollary compared with Cor. i. Prop. viii.

PROP. XII.

The fame Things being fuppofed as in the fecond Corollary of the viii[th] *Propofition.*

1. $m E^2 - \overline{m + 1} \times 2 D F + \overline{m + 4} \times 2 C G - \overline{m + 9} \times \left.\begin{matrix} \\ 2 B H + \overline{m + 16} \times 2 A I - \overline{m + 25} \times 2 K \end{matrix}\right\} = a'.$

2. $m' E^2 - 2 m' D F + \overline{m' - 12} \times 2 C G - \overline{m' - 72} \times \left.\begin{matrix} \\ 2 B H + \overline{m' - 240} \times 2 A I - \overline{m' - 600} \times 2 K \end{matrix}\right\} = a''.$

3. $m'' E^2 - 2 m'' D F + 2 m'' C G - \overline{m'' + 360} \times \left.\begin{matrix} \\ 2 B H + \overline{m'' + 360} \times 8 \times 2 A I - \overline{m'' + 360} \times 35 \times 2 K \end{matrix}\right\} = a'''.$

4. $m''' \times E^2 - 2 D F + 2 C G - 2 B H + \overline{m''' - 750} \times 28 \times \left.\begin{matrix} \\ A I - \overline{m''' - 7200} \times 28 \times 2 K \end{matrix}\right\} = a''''.$

&c.

Thefe Theorems follow from the third Corollary of the vi[th] Propofition compared with the fecond Corollary of the eighth Propofition. The firft is the fame with the firft of the laft Propofition. The fecond is demonftrated by fubftituting in $m' E' E' - 24 C' G' = a''.$ The Values of $E' E'$ and $C' G'$ given in the third

Cor.

Cor. of the vith Propofition. The third is found by fubftituting in $m\prime\prime\, E\prime E\prime - 720\, B\prime H\prime = a\prime\prime\prime$ the Values of $E\prime E\prime$ and $B\prime H\prime$; and by a like Subftitution thefe Theorems may be continued.

A General C O R O L L A R Y.

From thefe Propofitions a great Variety of Rules may be deduced for difcovering when an Equation has imaginary Roots. The Foundation of Sir *Ifaac New-ton*'s Rule is demonftrated in the ninth Propofition, and its Corollary. The feventh Propofition fhews that if

$$\frac{l-1}{2\,l} \times E^2 \text{ does not exceed } D F - C G + B H - A I$$

$+$ K, fome of the Roots of the Equation muft be imaginary; and fometimes this Rule will difcover impoffible Roots in an Equation, that do not appear by Sir *Ifaac Newton*'s Rule. Thefe are the only two Rules that have been hitherto publifhed. But the Rules that arife from the Theorems in the eleventh and twelfth Propofitions, are preferable to both; becaufe any imaginary Roots that can be difcovered by the viith or ixth always appear from the xith and xiith Propofitions; and impoffible Roots will often be difcovered by the xith and xiith Propofitions in an Equation, that do not appear in that Equation when examined by the viith and ixth Propofitions. The Advantage which the Rules deduced from the xith Propofition, have above thofe deduced from the preceeding Propofitions, will be manifeft by confidering that in the xith Propofition we have the Values of the Quantities $a\prime, b\prime, c\prime, d\prime, e\prime$, feparately; whereas in the preceeding Propofitions, we have only the Values of certain Aggregates of thefe Quantities

joined

joined with the fame Signs. Now it is obvious that if thefe Quantities be feparately found pofitive, any fuch Aggregates of them muft be pofitive; but thefe Aggregates may be pofitive, and yet fome of the Quantities a', b', c', d', e', themfelves may be found negative: From which it follows, that if the Roots of the Equation are all affected with the fame Sign, and no impoffible Roots appear by Propofition xi^{th}, none will appear by the preceeding Propofitions; but that fome imaginary Roots may be difcovered by Propofition xi^{th}, when none appear in the Equation examined by the Propofitions that preceed the xi^{th}. If fome of the Roots of the Equation are pofitive, and fome negative (which always eafily appears by confidering the Signs of the Terms of the Equation) then the xii^{th} Propofition will be in many Cafes more apt to difcover imaginary Roots in an Equation than thofe that preceed it.

The Rule that flows from the firft Theorem of the xi^{th} Propofition, obtains when the Roots of the Equation are affected with different Signs, as well as when they all have the fame Sign, and it is this; Multiply the Number of the Terms in an Equation that preceeds any Term, as $E x^{n-r}$ by the Number of Terms that follow it in the fame Equation, and call the Product m. Suppofe that $+D$, $-C$, $+B$, $-A$, $+I$ are the Coefficients preceeding the Term $E x^{n-r}$, and that $+F$, $-G$, $+H$, $-I$, $+K$ are the Coefficients that follow it; then if $\frac{1}{2}m E^2$ does not exceed $\overline{m+1} \times D F$ $- \overline{m+4} \times C G + \overline{m+9} \times B H - \overline{m+16} \times A I + \overline{m+25} \times K$ the Equation muft have fome imaginary Roots; where the Coefficients $m+1$, $m+4$, $m+9$, &c.

&c. are found by adding to m the Squares of the Numbers 1, 2, 3, 4, &c. which shew the Diſtances of the Coefficients to which they are prefixed, from the Coefficient E. The ſecond Theorem of the xiith Propoſition ſhews, that if $\frac{1}{2}m'\,E^2$ does not exceed $m'\,D\,F$

$$- \overline{m' - 12} \times C\,G + \overline{m - 72} \times B\,H - \overline{m' - 240}$$
$$\times A\,I + \overline{m' - 600} \times K,$$ the Equation muſt have ſome Roots imaginary.

For an Example, If the four Roots of the Biquadratick Equation $x^4 - A\,x^3 + B\,x^2 - C\,x + D = 0$ are real Quantities, it will follow equally from the vth, viith, ixth, and xith Propoſitions, that $\frac{3}{8}A^2$ muſt be greater than B, and that $\frac{3}{8}C^2$ muſt exceed B D. The viith further ſhews that $\frac{5}{12}B^2$ muſt exceed $A\,C - D$; the ixth demonſtrates that $\frac{4}{9}B^2$ muſt exceed A C; but our Rule deduced from Prop xi. ſhews that $2\,B^2$ muſt exceed $5\,A\,C - 8\,D$, the exceſs being $\frac{1}{2}a'$, and the Rule deduced from the ſecond Theorem of the xiith Propoſition ſhews that B^2 muſt always exceed $2\,A\,C + 4\,D$, the Exceſs being $\frac{1}{4}a''$. It appears from ſeveral preceeding Propoſitions, that if the Roots of the Equation have all the ſame Sign, then A C muſt exceed 16 D : Let the Exceſſes $5\,B^2 - 12\,A\,C + 12\,D = p$, $4\,B^2 - 9\,A\,C = q$, $A\,C - 16\,D = s$; and it is plain that $a'\,(= 4\,B^2 - 10\,A\,C + 16\,D) = q - s$

$- s = \dfrac{2}{5} \times \overline{2p - s}$; and that $a'' = q + s = \dfrac{2}{5} \times$

$\overline{2p + 4s}$. Let us fuppofe,

1. That s is pofitive, then it is manifeft that if either p or q be negative, a' muft alfo be found negative, and confequently that when the viith or ixth Propofitions fhew any Roots to be imaginary, the xith Propofition muft difcover them at the fame time. But as a'

$(= q - s = \dfrac{2}{5} \times \overline{2p - s})$ may be found negative

when p and q are both pofitive, it follows that the Rule we have deduced from the xith Propofition may difcover imaginary Roots in an Equation, that do not appear by the preceeding Propofitions: Thus if you examine the Equation $x^4 - 6\,x^3 + 10\,x^2 - 7\,x + 1$ by Sir *Ifaac Newton*'s Rule, or by our viith Propofition, no imaginary Roots appear in it from

either. But fince $2\,B^2 - 5\,AC + 8\,D\ (= \dfrac{1}{2} a') =$

$200 - 210 + 8 = - 2$ is in this Equation negative, it is manifeft that two Roots of the Equation muft be imaginary. Let us fuppofe

2 That s is negative, and that from the Signs of the Terms of the Equation, it appears that fome Roots are pofitive and fome negative; then in Order to fee if the Equation has any imaginary Roots, the moft ufeful Rule is that we deduced from the fecond Theorem of Prop. xii. *viz.* that if B^2 does not exceed $2\,AC + 4\,D$ fome of the Roots of the Equation muft be imaginary: For the Excefs of B^2 above $2\,AC + 4\,D$ being $\dfrac{1}{4} a'' = \dfrac{1}{4} \times \overline{q + s} = \dfrac{1}{10} \times \overline{2p + 4s}$, and s

being

being negative, it is manifeſt, that if q or p be negative $\frac{1}{4}\, a''$ muſt be negative; and that $\frac{1}{4}\, a''$ may be negative when q and p are both poſitive; that is, This Rule muſt always diſcover ſome Roots to be imaginary when the viith or ixth Propoſitions diſcover any impoſſible Roots in an Equation; and will very often diſcover ſuch Roots in an Equation when theſe Propoſitions diſcover none. For Example, if you examine the Equation $x^4 + 5x^3 + 6x^2 - x - 12 = 0$, you will diſcover no imaginary Roots in it by the viith or ixth Propoſitions; and though $AC - 16\,D\ (= s)$ be negative, it does not follow, that the Equation has any impoſſible Roots, becauſe it appears from the Signs of the Terms, that the Equation has Roots affected with different Signs. But ſince $B^2 - 2\,AC - 4\,D\ (= 36 + 10 - 48 = -2)$ is negative, it appears from our Rule, that the Equation muſt have ſome imaginary Roots.

I might ſhew in the next Place, how the Rules deduced from the xith and xiith Propoſitions may be extended ſo as to diſcover when more than two Roots of an Equation are imaginary, and in general to determine the Number of imaginary Roots in any Equation; but as it would require a long Diſcuſſion, and ſome *Lemmata* to demonſtrate this ſtrictly, I ſhall only obſerve that theſe xith and xiith Propoſitions will be found to be ſtill the moſt uſeful of all thoſe we have given for that Purpoſe. To give one Example of this; If we are to examine the Equation $x^4 - 4\,a\,x^3 + 6\,a^2\,x^2 - 4\,a\,b^2\,x + b^4 = 0$ by Sir *Iſaac Newton*'s Rule, it is found to have four impoſſible Roots when a is greater than b; for though the Square of the ſecond Term multiplied

ed by $\dfrac{3}{8}$ be equal to the Product of the firſt and third Terms, yet in that Caſe, in applying Sir *Iſaac New-ton's* Rule, the Sign — ought to be placed under the ſecond Term, and the ſame is to be ſaid of the Square of the fourth Term. The Rule deduced from the vii$^{\text{th}}$ Propoſition ſhews four Roots imaginary, when a is greater than b, and alſo when b^2 is greater than $15\ a^2$; but a Rule founded on the xi$^{\text{th}}$ Propoſition, ſhews the four Roots to be imaginary always when a exceeds b, or when b^2 exceeds $9\ a^2$; from which the Excellency of this Rule above theſe two is manifeſt. I have ſaid ſo much of Biquadratick Equations, that I muſt leave it to thoſe that are willing to take the Trouble, to make like Remarks on the higher Sorts of Equations.

In inveſtigating the preceeding Propoſitions, when I found my ſelf obliged to go through ſo intricate Cal-culations, I often attempted to find ſome more eaſy Way of treating this Subject. The following was of conſiderable Uſe to me, and may perhaps be entertaining to you. By it, I inveſtigate ſome *maxima* in a very ea-ſy Manner, that could not be demonſtrated in the com-mon Way with ſo little Trouble.

LEMMA V. Let the given Line A B be divided any where in P and the Rectangle of the Parts A P and P B will be a *maximum* when theſe Parts are e-   A————————P————B
qual.
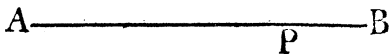
This is manifeſt from the Elements of *Euclid*.

LEMMA VI. If the Line A B is divided into any Number of Parts A B, C D, D E, E B, the Product of all thoſe Parts multiplied into one another will be a

max-

maximum when the Parts are equal amongſt themſelves. For let the Point D be where you will, it is manifeſt that if D B be biſſected in E, the Product A C × C D × D E × E B will be

greater than A C × C D × D e × e B

A ——————— C ——— D ——— E —— *e* —— B

becauſe by the laſt Lemma D E × E B is greater than D *e* × *e* B ; and for the ſame reaſon A D and C E muſt be biſſected in C and D ; and conſequently all the Parts A C, C D, D E, E B muſt be equal amongſt themſelves, that their Product may be a *maximum*.

L E M M A VII. The Sum of the Products that can be made by multiplying any two Parts of A B by one another is a *maximum* when the Parts are equal. The Sum of theſe Products is A C × C B + C D × D B + D E × E B : Now that D E × E B may be a *maximum*, D B muſt be biſſected in E by the vᵗʰ Lemma, and for the ſame reaſon A D and C E muſt be biſſected in C and D, that is all the Parts, A C, C D, D E, E B muſt be equal, that the Sum of all theſe Products may be a *maximum*.

L E M M A VIII. The Sum of the Products of any three Parts of the Line A B is a *maximum*, when all the Parts are equal. For that Sum is A C × C D × D E + E B × A C × C D + A C × D E + C D × D E ; and ſuppoſing the Point E given, it is manifeſt that A E muſt be equally trifected in C and D that A C × C D × D E may be a *maximum* by Lemma vi. and that A C × C D + A C × D E + C D × D E may be a *maximum* by Lemma viiᵗʰ. From which it is manifeſt that all the Parts A C, C D, D E, E B muſt be equal, that the Sum of the Products of any three of them may be a *maximum*.

L E M M A IX. It is manifeſt that this way of rea-ſoning is general, and that the Sum of any Quantities being given, the Sum of all the Products that can be

M   made

made by multiplying any given Number of them by one another, muſt be a *maximum* when theſe Quantities are equal. But the Sum of the Squares, or of any pure Powers of theſe Quantities, is a *minimum*, when the Quantities are equal.

THEOREM.

Suppoſe $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} - E x^{n-5}$ &c. $= 0$, *to be an Equation that has not all its Roots equal to one another: Let r expreſs the Dimenſions of any Coefficient* D, *and let*

$$l = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \text{ &c. } taking \ as \ ma\text{-}$$

ny Factors as there are Units in r; then ſhall $\dfrac{l}{n^r} \times A^r$

be always greater than D, *if the Roots of the Equation are real Quantities affected with the ſame Sign.*

This may be demonſtrated from the preceeding Propoſitions: But to demonſtrate it from the laſt Lemmata, let us aſſume an Equation that has all its Roots equal to one another, and the Sum of all its Roots equal to A, the Sum of the Roots of the propoſed Equa-

tion. This Equation will be $\overline{x - \dfrac{1}{n} A}\,|^n = 0$, or

$$x^n - A x^{n-1} + n \times \frac{n-1}{2} \times \frac{A^2}{n^2}\, x^{n-2} - n \times$$

$$\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{A^3}{n^3} x^{n-3} \text{ &c. } = 0 \text{ and if } r \text{ ex-}$$

preſs the Dimenſions of the Coefficient of any Term of this Equation (or the Number of Terms which

pre-

preceed it) it is manifeſt that the Term it felf will be $l \times \dfrac{A^r}{n^r} \, x^{n-r}$: But by the Suppoſition $D \, x^{n-r}$ is the

Correſponding Term in the propoſed Equation, and D muſt be the Sum of all the Products that can be made by multiplying as many Roots of that Equation by one another, as there are Units in r; and $\dfrac{l \, A^r}{n^r}$ muſt be the Sum of the like Products of the Roots of the other Equation ; which muſt be the greater Quantity by the preceeding Lemmata, becauſe its Roots are equal amongſt themſelves, and their Sum is equal to the Sum of the Roots of the propoſed Equation; and the Sum of ſuch Products is a *maximum* when the Roots are equal amongſt themſelves. By purſuing this Method,

it may be demonſtrated that $\left|\overline{\dfrac{2B}{n \times \overline{n-1}}}\right|^{\frac{r}{2}} \times l$ muſt always

exceed the Coefficient prefixed to the Term x^{n-r} in an Equation whoſe Roots are all real Quantities affected with the ſame Sign ; providing that r be a Number

greater than 2 ; and alſo that $\left|\overline{\dfrac{2 \times 3 \, C}{n \times \overline{n-1} \times \overline{n-2}}}\right|^{\frac{r}{3}} \times l$

muſt exceed the ſame Coefficient, if r be any Number greater than 3.

 It is eaſy to continue theſe Theorems.

 The third Method which I mentioned in the Beginning of this Letter, is deduced from the Conſideration of the Limits of the Roots of Equations; and though it is explained by ſome Authors already, yet as I de-

monſtrate

monſtrate and apply it to this Subject in a different Manner, I ſhall add a ſhort Account of it.

LEMMA X. If you transform the Biquadratick $x^4 - A x^3 + B x^2 - C x + D = 0$ into one that ſhall have each of its Roots leſs than the reſpective Values of x by a given Difference e; ſuppoſe $y = x - e$ or $x = e + y$ and the transformed Equation, the Order of the Terms being inverted, will have this Form.

$$e^4 + 4 \quad e^3 y + 6 e^2 \quad y^2 + 4 e y^3 + y^4 = 0.$$
$$- A e^3 - 3 A e^2 y - 3 A e y^2 - A y^3$$
$$+ B e^2 + 2 B e \; y + \quad B \; y^2$$
$$- C e - \quad c \; y$$
$$+ D$$

Where it is manifeſt,

1. That the firſt Term $e^4 - A e^3 + B e^2 - C e + D$ is the Quantity that ariſes by ſubſtituting e in Place of x in the propoſed Equation $x^4 - A x^3 + B x^2 - C x + D$.

2. That the Coefficient of the ſecond Term $4 e^3 - 3 A e^2 + 2 B e - C$ is the Quantity that ariſes by multiplying each Part of the firſt $e^4 - A e^3 + B e^2 - C e + D$ by the Index of e in that Part, and dividing the Product by e.

3. That the Coefficient of the third Term $6 e^2 - 3 A e + B$ is the Quantity that ariſes from the preceeding Coefficient $4 e^3 - 3 A e^2 + 2 B e - C$ by multiplying each Part by the Index of e in it, and dividing the Product by $2 e$.

4. That the Coefficient of the fourth Term ariſes in like Manner from the preceeding, only you now divide by $3 e$; and in general, the Coefficient of any Term may be deduced from the Coefficient of that Term which preceeds it, by multiplying each Part of

2   the

the preceeding Coefficient by the Index of e in that Part, and dividing the Product by e and by the Index of y, in the Term whofe Coefficient is required.

LEMMA XI. If any Equation $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3}$ &c. $= 0$ be transformed in the fame Manner, by fuppofing $x = y - e$ or $x = e + y$, and confequently $x^n = \overline{e+y}\,|^n$, $A x^{n-1} = A \times \overline{e+y}\,|^{n-1}$, $B x^{n-2} = B \times \overline{e+y}\,|^{n-2}$ &c. The transformed Equation will have this Form, the Order of the Terms being inverted,

$$e^n + n e^{n-1} y + n \times \frac{n-1}{2} \times e^{n-2} y^2 \text{ \&c.} = 0$$

$$- A e^{n-1} - \overline{n-1} \times A e^{n-2} y \quad - \overline{n-1} \times \frac{n-2}{2} \times A e^{n-3} y^2 \text{ \&c.}$$

$$+ B e^{n-2} + \overline{n-2} \times B e^{n-3} y \quad + \overline{n-2} \times \frac{n-3}{2} \times B e^{n-4} y^2 \text{ \&c.}$$

$$- C e^{n-3} - \overline{n-3} \times C e^{n-4} y \quad - \overline{n-3} \times \frac{n-4}{2} \times C e^{n-5} y^2 \text{ \&c.}$$

&c. &c. &c.

Where it is manifeft,

1. That the firft Term $e^n - A e^{n-1} + B e^{n-2} - C e^{n-3}$ &c. is the Quantity that arifes by fubftituting e in the Place of x in the propofed Equation $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3}$ &c.

2. That the Coefficient of the fecond Term $n e^{n-1} - \overline{n-1} \times A e^{n-2} + \overline{n-2} \times B e^{n-3} - \overline{n-3} \times C e^{n-4}$ &c. is deduced from the preceeding $e^n - A e^{n-1} + B e^{n-2} - C e^{n-3}$ &c. by multiplying each of its Parts by the Index of e in that Part, and dividing by e.

3. That

(84)

3. That the Coefficient of the third Term is deduced from the Coefficient of the fecond Term, by multiplying after the fame manner, each of its Parts by the Index of e and dividing by 2 e. In general, the Coefficient of any Term y^r is deduced from the Coefficient of the preceeding Term, that is of y^{r-1} by multiplying every Part of that Coefficient by the Index of e in it, and dividing the Product by $r\,e$.

L E M M A XII. If you fubftitute any two Quantities K and L in the Place of x in $x^4 - A x^3 + B x^2 - C x + D$, and the Quantities that refult from thefe Subftitutions be affected with contrary Signs, the Quantities K and L muft be *Limits* of one or more real Roots of the Equation $x^4 - A x^3 + B x^2 - C x + D = 0$. That is, one of thefe Quantities muft be greater, and the other lefs than one or more Roots of that Equation.

For if you fuppofe that a, b, c, d, are the Roots of that Equation, then it is plain from the *Genefis* of Equations, that $x^4 - A x^3 + B x^2 - C x + D = \overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$; and therefore K and L being fubftituted for x in $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, the Product becomes in the one Cafe pofitive, and in the other negative; fo that one of the Factors $x-a, x-b, x-c, x-d$ muft have a Sign when K is fubftituted for x in it, contrary to the Sign which it is affected with when L is fubftituted in in it for x, fuppofe that Factor to be $x-b$; and fince $K-b$ and $L-b$ are Quantities whereof the one is pofitive, and the other negative, it is manifeft that b one of the Roots of the Equation muft be lefs than one, and greater than the other of the two Quantities

3   tities

tities K and. L : So that K and L muſt be the *Limits* of the Root *b.*

I ſay further, that the Root whereof K and L are *Limits,* muſt be a real Root of the Equation ; for the Product of the Factors that involve impoſſible Roots in an Equation can never have its Signs changed by ſub-ſtituting any real Quantity whatſoever in place of *x* ; becauſe the Number of ſuch Roots is always an even Number, and the Product of any two of theſe Roots ſuch as $x - m - \sqrt{-n}$, and $x - m + \sqrt{-n}$ is $\overline{x - m}|^2 + n^2$ which muſt be always poſitive, what-ever Quantity be ſubſtituted for *x* while *n* remains poſitive, that is, while theſe two Roots are impoſſible.

LE MM A XIII. If you ſubſtitute K and L for *x* in $x^n - A x^{n-1} + B x^{n-2}$ &c. and the Quantities that reſult be affected with contrary Signs, then ſhall K and L be the *Limits* of one or more real Roots of the Equation $x^n - A x^{n-1} + B x^{n-2}$ &c. $= 0.$ This may be demonſtrated after the ſame Manner as the laſt Lemma.

THEOREM I. If a, b, c, d are the Roots of the Equation $x^4 - A x^3 + B x^2 - C x + D = 0,$ they ſhall be the *Limits* of the Roots of the Equation $4 x^3 - 3 A x^2 + 2 B x - C = 0.$

Suppoſe *a* to be the leaſt Root of the biquadratick $x^4 - A x^3 + B x^2 - C x + D = 0,$ *b* the ſecond Root, *c* the third, and *d* the fourth, and the Values of *y* in the Equation in the xth Lemma, will be $a - e,$ $b - e, c - e, d - e$; then by ſubſtituting ſucceſſive-ly a, b, c, d for *e* in that Equation of *y*, one of the Values of *y* will vaniſh in every Subſtitution, and the firſt Term of the Equation of *y*, *viz.* $e^4 - A e^3 + B e^2 - C e + D$ vaniſhing, the Equation will be re-duced to a Cubick of this Form.

4 *e*

$$4e^3 + 6e^2y + 4ey^2 + y^3 = 0$$
$$- 3Ae^2 - 3Aey - Ay^2$$
$$+ 2Be + By$$
$$- C$$

And confequently $4e^3 - 3Ae^2 + 2Be - C$ muft be the Product of the three remaining Values of y having its Sign changed ; that is, it muft be equal to $-\overline{b-a} \times \overline{c-a} \times \overline{d-a}$ when e is fuppofed equal to a, it muft be $-\overline{a-b} \times \overline{c-b} \times \overline{d-b}$ when $e = b$; it muft be $-\overline{a-c} \times \overline{b-c} \times \overline{d-c}$ when $e = c$; and it muft be $-\overline{a-d} \times \overline{b-d} \times \overline{c-d}$ when $c = d$. Now it is manifeft that thefe Products $\overline{b-a} \times \overline{c-a} \times \overline{d-a}$, $\overline{a-b} \times \overline{c-b} \times \overline{d-b}$, $\overline{a-c} \times \overline{b-c} \times \overline{d-c}$, $\overline{a-d} \times \overline{b-d} \times \overline{c-d}$ muft be affected with the Signs $+, -, +, -$ refpectively; the firft being the Product of three pofitive Quantities, the fecond the Product of one negative and two pofitives, the third the Product of two negatives and one pofitive, and the fourth the Product of three negatives. Therefore fince by fubftituting a, b, c, d for e in the Quantity $4e^3 - 3Ae^2 + 2Be - C$, it becomes alternately a pofitive and a negative Quantity, it follows from the laft Lemma that a, b, c, d muft be the *Limits* of the Roots of the Equation $4e^3 - 3Ae^2 + 2Be - C = 0$, or of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$.

Cor. It follows from this Theorem, that if $a' b'$ and c' are the three Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$, they muft be *Limits* betwixt a, b, c, d the Roots of the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$: For if a, b, c, d are *Limits* of the
Roots

Roots a', b', and c'; thefe Roots converfely muft be *Limits* betwixt a, b, c and d.

THEOREM II. Multiply the Terms of any Biquadratick $x^4 - A x^3 + B x^2 - C x + D = 0$ by any Arithmetical Series of Quantities $l + 4m, l + 3m, l + 2m, l + m, l,$ and the Roots of the Biquadratick a, b, c, d will be the *Limits* of the Roots of the Equation that refults from that Multiplication that is of the Equation.

$$l x^4 - l A x^3 + l B x^2 - l C x + l D = 0$$
$$+ 4 m x^4 - 3 m A x^3 + 2 m B x^2 - m C x$$

Suppofe that fubftituting the Roots a, b, c, d of the biquadratick Equation $x^4 - A x^3 + B x^2 - C x + D = 0$ fucceflively, for x in $4 x^3 - 3 A x^2 + 2 B x - C$, the Quantities that refult are $- R, + S, - T, + Z$; while $x^4 - A x^3 + B x^2 - C x + D$ is in every Subftitution equal to nothing; and it is manifeft that the Quantity

$$+ l x^4 - l A x^3 + l B x^2 - l C x + l D$$
$$+ 4 m x^4 - 3 m A x^3 + 2 m B x^2 - m C x$$

will become (when a, b, c, d are fubftituted fucceflively in it for x) equal to $- m R x, + m S x, - m T x, + m Z x$; where the Signs of thefe Quantities being alternately negative and pofitive, it follows that a, b, c, d muft be *Limits* of that Equation by Lemma xii.

COR. Hence it follows, that a, b, c and d are *Limits* of the Roots of the Cubick Equation $A x^3 - 2 B x^2 + 3 C x - 4 D = 0$, and converfely, that the Roots of this Cubick are *Limits* of the Roots of the biquadratick Equation $x^4 - A x^3 + B x^2 - C x + D = 0$, for multiplying the Terms of this biquadratick Equation by the Arithmetical Progreflion $0, - 1, - 2, - 3, - 4,$ the Cubick $A x^3 - 2 B x^2 + 3 C x - A D = 0$ arifes.

N   THE-

THEOREM III. *In general, the Roots of the Equa-tion* $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3}$ &c. $= 0$, *are the* Limits *of the Roots of the Equation* $n x^{n-1} - \overline{n-1} \times A x^{n-2} + \overline{n-2} \times B x^{n-3}$ &c. $= 0$, *or of any Equation that is deduced from it by multi-plying its Terms by any Arithmetical Progreſſion* $1 \mp d, 1 \mp 2 d, 1 \mp 3 d$ &c. *and converſely the Roots of this new Equation will be the* Limits *of the Roots of the propoſed Equation* $x^n - A x^{n-1} + B x^{n-2}$ &c. $= 0$.

This Theorem is demonſtrated from the xi[th] and xiii[th] Lemmata in the ſame manner as the preceeding Theorems were demonſtrated from the xth and xii[th]. From theſe Theorems it is eaſy to infer all that is deli-vered by the Writers of Algebra on this Subject.

THEOREM IV. *The Equation* $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3}$ &c. $= 0$ *will have as many ima-ginary Roots as the Equation* $n x^{n-1} - \overline{n-1} \times A x^{n-2} - \overline{n-2} \times B x^{n-3}$ &c. $= 0$, *or the Equa-tion* $A x^{n-1} - 2 B x^{n-2} + 3 C x^{n-3}$ &c. $= 0$.

Suppoſe that any Root of the Equation $n x^{n-1} - \overline{n-1} \times A x^{n-2} + \overline{n-2} \times B x^{n-3}$ &c. $= 0$, as p becomes imaginary, and the two Roots of the Equa-tion $x^n - A x^{n-1} + B x^{n-2}$ &c $= 0$, which by Theorem III. ought to be its *Limits*, cannot both be real Quantities; for it is manifeſt from the Demonſtra-tion of Theorem I. that if they are real Quantities, then being ſubſtituted for x in $n x^{n-1} - \overline{n-1} \times A x^{n-2} + \overline{n-2} \times B x^{n-3}$ &c. the Quantities that reſult muſt have contrary Signs, and conſequently the Root p, whereof they are *Limits*, muſt be a real Root ; which

which is againſt the Suppoſition. The ſame is true of the Equation $A x^{n-1} - 2 B x^{n-2} + 3 C x^{n-3}$ &c. $= 0$, for the ſame Reaſon.

Cor. The biquadratick $x^4 - A x^3 + B x^2 - C x + D = 0$, will have two imaginary Roots, if two Roots of the Equation $4 x^3 - 3 A x^2 + 2 B x - C = 0$ be imaginary; or if two Roots of the Equation $A x^3 - 2 B x^2 + 3 C x - 4 D = 0$ be imaginary. But two Roots of the Equation $4 x^3 - 3 A x^2 + 2 B x - C = 0$ muſt be imaginary, when two Roots of the Quadratick $6 x^2 - 3 A x + B = 0$, or of the Quadratick $3 A x^2 - 4 B x + 3 C = 0$ are imaginary, becauſe the Roots of theſe quadratick Equations are the *Limits* of the Roots of that Cubick, by the third Theorem ; and for the ſame reaſon two Roots of the Cubick Equation $A x^3 - 2 B x^2 + 3 C x - 4 D = 0$ muſt be imaginary, when the Roots of the quadratick $3 A x^2 - 4 B x + 3 C = 0$, or of the quadratick $B x^2 - 3 C x + 6 D = 0$ are impoſſible. Therefore two Roots of the Biquadratick $x^4 - A x^3 + B x^2 - C x + D = 0$ muſt be imaginary when the Roots of any one of theſe three quadratick Equations $6 x^2 - 3 A x + B = 0$, $3 A x^2 - 4 B x + 3 C = 0$, $B x^2 - 3 C x + 6 D = 0$ become imaginary; that is, when $\frac{3}{8} A^2$ is leſs than B, $\frac{4}{9} B^2$ leſs than A C, or $\frac{3}{8} C^2$ leſs than B D.

Cor. II. By proceeding in the ſame manner, you may deduce from any Equation $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3}$ &c. $= 0$, as many quadratick Equations as there are Terms excepting the firſt and laſt whoſe Roots muſt be all real Quantities, if the

pro-

propofed Equation has no imaginary Roots. The Quadratick deduced from the three firſt Terms x^n — $A x^{n-1}$ + $B x^{n-2}$ will manifeſtly have this Form,

$$\overline{n \times n - 1} \times \overline{n - 2} \times \overline{n - 3} \ \&c. \times x^2 - \overline{n - 1} \times$$

$$\overline{n - 2} \times \overline{n - 3} \times \overline{n-4} \ \&c. \times A x + \overline{n-2} \times \overline{n - 3}$$

$\times \overline{n - 4} \times \overline{n - 5} \ \&c. \times B = 0$, continuing the Factors in each till you have as many as there are Units in $n - 2$. Then dividing the Equation by all the Factors $n - 2$, $n - 3$ &c. which are found in each Coefficient, the Equation will become $n \times \overline{n - 1} \times x^2 -$ $\overline{n - 1} \times 2 A x + 2 \times 1 \times B = 0$, whoſe Roots will be imaginary by Prop. i. when $n \times \overline{n - 1} \times 2 \times 4 B$ exceeds $\overline{n - 1}|^2 \times 4 A^2$, or when B exceeds $\dfrac{n - 1}{2 n} A^2$, ſo that the propoſed Equation muſt have ſome imaginary Roots when B exceeds $\dfrac{n - 1}{2 n} A^2$; as we demonſtrated after another Manner in the vth Propoſition. The Quadratick Equation deduced in the ſame Manner from the three firſt Terms of the Equation $A x^{n-1} - 2 B x^{n-2}$ + $3 C x^{n-3}$ &c. $= 0$, will have this Form $\overline{n - 1} \times$ $\overline{n - 2} \times \overline{n - 3} \&c. \times A x^2 - \overline{n - 2} \times \overline{n - 3} \times \overline{n - 4}$ &c. $\times 2 B x + \overline{n - 3} \times \overline{n - 4} \times \overline{n - 5}$ &c. $\times 3 C =$ 0 ; which by dividing by the Factors common to all the Terms, is reduced to $\overline{n - 1} \times \overline{n - 2} \times A x^2 - \overline{n - 2} \times$ $4 B x + 6 C = 0$, whoſe Roots muſt be imaginary when $\dfrac{2}{3} \times \dfrac{n - 2}{n - 1} \times B^2$ is leſs than $A C$; and therefore in that caſe ſome Roots of the propoſed Equation muſt be imaginary.

Cor. III. In general, let $D x^{n-r+1} - E x^{n-r}$ + $F x^{n-r-1}$ be any three Terms of the Equation, $x^n -$

$A x$

$A x^{n-1} + B x^{n-2}$ &c. $= 0$, that immediately follow one another, multiply the Terms of this Equation first by the Progreſſion $n, n-1, n-2$, &c. then by the Progreſſion $n-1, n-2, n-3$, &c. then by $n-2, n-3, n-4$, &c. till you have multiplied by as many Progreſſions as there are Units in $n-r-1$: Then multiply the Terms of the Equation that ariſes, as often by the Progreſſion $0, 1, 2, 3$ &c. as there are Units in $r-1$, and you will at length arrive at a Quadratick of this Form,

$$\overline{n-r+1} \times \overline{n-r} \times \overline{n-r-1} \times \overline{n-r-2} \, \&c. \times \overline{r-1}$$
$$\times \overline{r-2} \times \overline{r-3} \times \overline{r-4} \, \&c. \; D x^2$$

$$-\overline{n-r} \times \overline{n-r-1} \times \overline{n-r-2} \times \overline{n-r-3} \, \&c.$$
$$\times r \times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \, \&c. \times E x$$

$$+\overline{n-r-1} \times \overline{n-r-2} \times \overline{n-r-3} \times \overline{n-r-4} \&c.$$
$$\times \overline{r+1} \times r \times \overline{r-1} \times \overline{r-2} \, \&c. \times F = 0,$$

and dividing by the Factors $n-r-1, n-r-2$, &c. and $r-1, r-2$ &c. which are found in each Coefficient, this Equation will be reduced to $\overline{n-r+1}$ $\times \overline{n-r} \times 2 \times 1 \times D x^2 - \overline{n-r} \times 2 \times r \times 2 E x + 2 \times 1 \times \overline{r+1} \times r \; F = 0$, whoſe Roots muſt be imaginary (by Prop i.) when $\dfrac{n-r}{n-r+1} \times \dfrac{r}{r+1} \times E^2$ is leſs than $D F$. From which it is manifeſt that if you divide each Term of this Series of Fractions $\dfrac{n}{1}, \dfrac{n-1}{2},$

$\dfrac{n-2}{3}, \dfrac{n-3}{4},$ &c. $\dfrac{n-r+1}{r}, \dfrac{n-r}{r+1}$ by that which preceeds it, and place the Quotients above the Terms of the Equation $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3}$

&c. $=$

&c. $= 0$, beginning with the fecond: Then if the Square of any Term multiplied by the Fraction over it be found lefs than the Product of the adjacent Terms, fome of the Roots of that Equation muft be imaginary Quantities. There remain many things that might be added on this Subject, but I am afraid you will think I have faid as much of it as it deferves; and therefore I fhall only add the Demonftration of fome Algebraick Rules and Theorems that are very eafily deduced from the xith Lemma.

I. The Rule for difcovering when two or more Roots of an Equation are equal, immediately follows from that Lemma. Suppofe that two Roots of the Equation $x^n — A x^{n-1} + B x^{n-2} — C x^{n-3}$ &c. $= 0$ are equal, and two Values of y (which is equal always to $x—e$) will be equal. Suppofe that e is equal to one of thofe two equal Values of x; and two Values of y will vanifh, and confequently y^2 muft enter each of the Terms of the Equation of y; and therefore in this Cafe the firft and fecond Term of the Equation of y in Lemma xith muft vanifh, that is, $e^n — A e^{n-1} + B e^{n-2} — C e^{n-3}$ &c. $= 0$ and $n e^{n-1} — \overline{n-1} \times A e^{n-2} + \overline{n-2} \times B e^{n-3} — \overline{n-3} \times C e^{n-4}$ &c. $= 0$ at the fame time; and confequently thefe two Equations muft have one Root common, which muft be one of thofe Values of x that were fuppofed equal to each other. It is manifeft therefore that when two Values of x are equal in the Equation $x^n — A x^{n-1} + B x^{n-2}$ &c. $= 0$, one of them muft be a Root of the Equation $n x^{n-1} — \overline{n-1} \times A x^{n-2} + \overline{n-2} \times B x^{n-3}$ &c. $= 0$.

If three Values of x be fuppofed equal amongft themfelves and to e, then three Values of y ($= x — e$) will vanifh, and the firft three Terms of the Equation of y

in

in Lemma xi. will vaniſh, and therefore $n \times \overline{n-1}$ $\times e^{n-2} - \overline{n-1} \times \overline{n-2} \times A e^{n-3} + \overline{n-2} \times \overline{n-3}$ $\times B e^{n-4}$ &c. $= 0$; and one of the equal Values of x will be a Root of this laſt Equation, and two of them will be Roots of the Equation $n x^{n-1} - \overline{n-1} \times$ $A x^{n-2} + \overline{n-2} \times B x^{n-3}$ &c. $= 0$. In general, it appears that if the Equation $x^n - A x^{n-1} + B x^{n-2}$ &c. $= 0$ have as many Roots equal amongſt themſelves as there are Units in S, then ſhall as many of thoſe be Roots of the Equation $n x^{n-1} - \overline{n-1} \times A x^{n-2}$ $+ \overline{n-2} \times B x^{n-3}$ &c. $= 0$ as there are Units in S $-$ 1; as many of them ſhall be Roots of the Equation $n \times \overline{n-1} \times x^{n-2} - \overline{n-1} \times \overline{n-2} \times A x^{n-3} +$ $\overline{n-2} \times \overline{n-3} \times B x^{n-4}$ &c. $= 0$, as there are Units in S $-$ 2; and ſo on.

II. The general Rule which Sir *Iſaac Newton* has given in the *Article de limitibus Equationum* for find-ing a *Limit* greater than any of the Values of x im-mediately follows from the xi^th Lemma; for it is mani-feſt that if e be ſuch a Quantity as ſubſtituted in all the Coefficients of the Equation of y, *viz.* in $e^n - A e^{n-1}$ $+ B e^{n-2}$ &c. $n e^{n-1} - \overline{n-1} \times A e^{n-2} + \overline{n-2}$ $\times B e^{n-3}$ &c. $n \times \dfrac{\overline{n-1}}{2} \times e^{n-2} - \overline{n-1} \times \dfrac{\overline{n-2}}{2} \times$ $A e^{n-3} + \overline{n-2} \times \dfrac{\overline{n-3}}{2} \times B e^{n-4}$ &c. gives th\ Quantities that reſult all poſitive; then there being no Changes of the Signs of the Equation of y in this caſe, all its Values muſt be negative; and ſince y is always equal to $x - e$ it follows that e muſt be a greater Quan-tity than any of the Values of x; that is, it muſt be a

Limit

Limit greater than any of the Roots of the Equation $x^n - A x^{n-1} + B x^{n-2}$ &c. $= 0$.

III. From this xith Lemma fome important Theorems in the Method of *Series*, and of *Fluxions*, and the Refolution of Equations are demonftrated with great Facility; it is obvious that the Coefficient of the fecond Term of the Equation of y in that Lemma is the *Fluxion* of the firft Term divided by the *Fluxion* of e; the Coefficient of the third Term is the fecond *Fluxion* of that firft Term divided by $2 \dot{e}^2$; fuppofing e to flow uniformly. The third Term is the third *Fluxion* of the firft Term divided by $2 \times 3 \dot{e}^3$; and fo on. Therefore fuppofing $e^n - A e^{n-1} + B e^{n-2}$ &c. $= c$, the Equation for determining y will be $c + \dfrac{\dot{c}}{\dot{e}} y + \dfrac{\ddot{c}}{1 \times 2 \dot{e}^2} y^2$

$+ \dfrac{\dddot{c}}{1 \times 2 \times 3 \dot{e}^3} y^3$ &c. $= 0$; and hence, when e is near the true Value of x, Theorems may be deduced for approximating to y, and confequently to x, which is fuppofed equal to $y + e$.

IV. Let $A P (= x)$ be the Abfcifs and $P M (= z)$ the Ordinate of any Curve $B L M$; and fuppofe any other Abfcifs $A K = e$ and Ordinate $K L = c$, then

fhall $z (= P M) = c \mp \dfrac{\dot{c}}{\dot{e}} y + \dfrac{\ddot{c}}{2 \dot{e}^2} y^2 \mp \dfrac{\dddot{c}}{2 \times 3 \dot{e}^3} y^3$

$+ \dfrac{c}{2 \times 3 \times 4 \dot{e}^4} y^4$ &c.

For let z be fuppofed equal to any Series confifting of given Quantities, and the Powers of x, as to $A x^n + B x^r + C x^s$ &c. and fubftituting $e \mp y$ for x, we fhall find after the manner of the xith Lemma,

$z =$

$$z = A e^{n} \mp n A e^{n-1} y + n \times \frac{n-1}{2} \times A e^{n-2} y^2 \ \&c.$$

$$+ B e^{r} \mp r B e^{r-1} y + r \times \frac{r-1}{2} \times B e^{r-2} \times y^2 \&c.$$

$$+ C e^{s} \mp s C e^{s-1} y + s \times \frac{s-1}{2} \times C e^{s-2} y^2 \&c.$$

$$\&c. \qquad \&c. \qquad \&c.$$

But when $x = e$ then $z = c = A e^{n} + B e^{r} + C e^{s}$
&c. $\dot{c} = n A e^{n-1} \dot{e} + r B e^{r-1} \dot{e} + s C e^{s-1} \dot{e}$ &c.
$\ddot{c} = n \times \overline{n-1} \times A e^{n-2} \dot{e}^2 + r \times \overline{r-1} \times B e^{r-2} \dot{e}^2$
$+ s \times \overline{s-1} \times C e^{s-2} \dot{e}^2$ &c. and therefore $z = c \mp$

$$\frac{\dot{c}}{\dot{e}} y + \frac{\ddot{c}}{2 \dot{e}^2} y^2 \mp \frac{\ddot{c}}{2 \times 3 \dot{e}^3} y^3$$ &c. After the same

manner you will find that $c = z \pm \dfrac{\dot{z}}{\dot{x}} y + \dfrac{\ddot{z}}{2 \dot{x}^2} y^2$

$$\pm \frac{\ddot{z}}{2 \times 3 \dot{x}^3} y^3$$ &c. for $c = A e^{n} + B e^{r} + C e^{s}$ &c. $=$

$A \times \overline{x \pm y}^{n} + B \times \overline{x \pm y}^{r} + C \times \overline{x \pm y}^{s}$ &c. $= z \pm \dfrac{\dot{z}}{\dot{x}} y$
$+ \ddot{z}$

O

$+ \dfrac{\ddot{z}}{2x^2} y^2$ &c. The Area K L M P is equal to the Fluent of $z\dot{y}$ or of $c\dot{y}$, but

$$c\,y = z\dot{y} \pm \dfrac{\dot{z}}{x}y\dot{y} + \dfrac{\ddot{z}}{2x^2}y^2\dot{y} \pm \dfrac{\dddot{z}}{2\times 3\,x^3}y^2\dot{y}\ \&c.$$

and $z\dot{y} = c\dot{y} \mp \dfrac{\dot{c}}{e}y\dot{y} + \dfrac{\ddot{c}}{2e^2}y^2\dot{y} \mp \dfrac{\dddot{c}}{2\times 3\,e^2}y^3\dot{y}$ &c.

And confequently by finding the Fluents

$$\mathrm{KLMP} = cy \mp \dfrac{\dot{c}}{2e}y^2 + \dfrac{\ddot{c}}{2\times 3\,e^2}y^3 \mp \dfrac{\dddot{c}}{2\times 3\times 4\,e^3}y^4\ \&c.$$

$$\text{or KLMP} = zy \pm \dfrac{\dot{z}}{2x}y^2 + \dfrac{\ddot{z}}{2\times 3\,x^2}y^3 \pm \dfrac{\dddot{z}}{2\times 3\times 4\,x^3}y^4\ \&c.$$

This laft is the Theorem publifhed by the learned Mr. *Bernouilli* in the *Acta Lipfiæ* 1694. It is now high Time to conclude this long Letter; I beg you may accept of it as a Proof of that Refpect and Efteem with which

<div align="center">

I am,

S I R,

Your moft Obedient,

Moft Humble Servant,

Colin Mac Laurin.

</div>